

Effective action for the order parameter of the deconfinement transition of Yang-Mills theories

Holger Gies

Institut für theoretische Physik, Universität Tübingen,
72076 Tübingen, Germany

February 7, 2008

Abstract

The effective action for the Polyakov loop serving as an order parameter for deconfinement is obtained in one-loop approximation to second order in a derivative expansion. The calculation is performed in $d \geq 4$ dimensions, mostly referring to the gauge group $SU(2)$. The resulting effective action is only capable of describing a deconfinement phase transition for $d > d_{\text{cr}} \simeq 7.42$. Since, particularly in $d = 4$, the system is strongly governed by infrared effects, it is demonstrated that an additional infrared scale such as an effective gluon mass can change the physical properties of the system drastically, leading to a model with a deconfinement phase transition.

1 Introduction

As a prelude to a truly nonperturbative evaluation of the effective action of Yang-Mills theory, the one-loop effective action with all-order couplings to a specific background may provide a first glance at the up to now unknown ground state of the theory. Since the problem of confinement is supposed to be intimately related to the quest for the ground state, it is elucidating to investigate the response of several “confining vacuum candidates” on quantum fluctuations – even in a perturbative approximation. In this spirit, e.g., the famous Savvidy model [1], which favors a covariant constant magnetic field as ground state, has given rise to much speculation on the nature of the vacuum.

Since a useful description of confinement and the ground state should also exhibit the limits of their formation, it is natural to perform a study at finite temperature where a transition to a deconfined phase is expected (as is observed on the lattice). An order parameter for the deconfinement phase transition in pure gauge theory is given by the Polyakov loop [2, 3], i.e., a Wilson line closing around the compactified Euclidean time

direction:

$$L(x) = \frac{1}{N_c} \text{tr} T \exp \left(ig \int_0^\beta dx_0 A_0(x_0, x) \right), \quad (1)$$

where the period $\beta = 1/T$ is identified with the inverse temperature of the ensemble in which the expectation value of L is evaluated. T denotes time ordering, N_c the number of colors, and A_0 is the time component of the gauge field. The negative logarithm of the Polyakov loop expectation value can be interpreted as the free energy of a single static color source living in the fundamental representation of the gauge group [4]. In this sense, an infinite free energy associated with confinement is indicated as $\langle L \rangle \rightarrow 0$, whereas $\langle L \rangle \neq 0$ signals deconfinement.

Moreover, $\langle L \rangle$ measures whether center symmetry, a discrete symmetry of Yang-Mills theory, is realized by the ensemble under consideration [4]. Gauge transformations which differ at $x_0 = 0$ and $x_0 = \beta$ by a center element of the gauge group change L by a phase $e^{2\pi i n/N_c}$, n integer (but leave the action invariant); this implies that a center-symmetric ground state automatically ensures $\langle L \rangle = 0$, whereas deconfinement $\langle L \rangle \neq 0$ is related to the breaking of this symmetry.

Therefore, the effective action governing the behavior of L is of utmost importance, because it determines the state of the theory at a given set of parameters, such as temperature, fields, etc. While this scenario has successfully been established in lattice formulations [5], several perturbative continuum investigations have led to various results. In the continuum, it is convenient to work with the “Polyakov gauge”, which rotates the zeroth component A_0 of the gauge field into the Cartan subalgebra of $SU(N_c)$, $A_0 \rightarrow A_0$ (cf. Eq. (2) below); furthermore, the condition $\partial_0 A_0 = 0$ is imposed. Then, if the A_0 ground state of the system is known, L can immediately be read off from Eq. (1), which suggests calculating the effective action of an time-independent A_0 background field.

Several one-loop calculations exist in the literature: considering a pure constant A_0 background, Weiss [6] obtained an effective potential for the Polyakov loop preferring only the center-asymmetric ground states, i.e., the deconfined phase (see Eq. (40) below). Combining the Savvidy model of a covariant constant magnetic background field with the Polyakov loop background A_0 , Starinets, Vshivtsev and Zhukovskii [7] as well as Meisinger and Ogilvie [8] were able to demonstrate the existence of a confining center symmetric minimum for $\langle L \rangle$ at low temperature with a transition to the broken, i.e., deconfined phase for increasing temperature. However, this model still suffers from the instabilities caused by the gluon spin coupling to the magnetic field [8], a problem also plaguing the Savvidy model [9], although an additional A_0 background can in principle remove the problematic tachyonic mode in the gluon propagator for certain values of gA_0 and T . Perhaps the most promising approach was explored by Engelhardt and Reinhardt [10], who considered a spatially varying A_0 field and evaluated the effective action for A_0 in a gradient expansion to second order in the derivatives. The resulting action exhibits both phases, confinement and deconfinement, depending on the value of the temperature; in particular at low temperature, the spatial fluctuations of the Polyakov loop lower the action

when fluctuating around the center symmetric (confining) phase. The main drawback of this model is its nonrenormalizability. An explicit cutoff dependence remains; gauge and Lorentz invariance have been broken explicitly during the calculation. Nevertheless, the main lesson to be learned is that spatial variations of the Polyakov loop have to be taken into account while searching for an effective potential of the order parameter for the deconfinement phase transition.

The present work is devoted to an investigation of the Polyakov loop potential partly in the spirit of [10]; however, the treatment of the quantum fluctuations, the calculational techniques, and finally the results are quite different. In particular, we employ the background field method to keep track of the symmetries of the functional integral [11]. Unfortunately, the results are not as promising as those found in [10], since the simple picture for the deconfinement phase transition is not visible in the most stringent version of the model.

The paper is organized as follows: in Sec. 2, we define the model, clarify our notations, and perform a first analysis of possible scenarios. Section 3 outlines the calculation of the effective action to one loop using the proper-time method, particularly emphasizing the subtleties of the present problem; we work in $d \geq 4$ dimensions with gauge group $SU(N_c)$. The implications of our results are discussed in Sec. 4 for $SU(2)$; therein it is pointed out that the main features of the model depend strongly on the treatment of the infrared sector. Section 5 briefly demonstrates the latter point by introducing an additional infrared scale “by hand” (gluon mass), which changes the properties of the model drastically, now exhibiting a confining phase. We finally comment on our findings in Sec. 6.

One last word of caution: it is obvious from the very beginning that the one-loop approximation performed here is hardly appropriate for dealing with the strongly coupled gauge systems under consideration. In fact, the results presented below mostly represent an extreme extrapolation of perturbation theory to extraordinary large values of the coupling constant g without any reasonable justification. Nevertheless, besides being interesting in its own right, the model can serve as a starting point for more involved investigations. E.g., the renormalization group flow of the true effective action will coincide with the perturbative action at large momentum scales; hence, a detailed knowledge of the perturbative regime will be of use for checking nonperturbative solutions. Moreover, some of the technical results of the present calculation such as the form of the gluon propagator in a fluctuating A_0 background will be expedient for other problems as well.

2 The Model

The essence of the model under consideration is determined by the choice of the background field, which is treated as a classical field subject to thermal and quantum fluctuations. At the very beginning, we confine ourselves to quasi-abelian background fields, pointing into a fixed direction n^a in color space:

$$A_\mu := A_\mu^a T^a =: A_\mu n^a T^a, \quad n^2 = 1, \quad (2)$$

where $(T^a)^{bc} \equiv -if^{abc}$ denote the hermitean generators of the gauge group $SU(N_c)$ in the adjoint representation.

Now we are aiming at a derivative expansion in the time-like component¹ of A_μ ; such an expansion is usually justified by demanding that the derivatives be smaller than the characteristic mass scale of the theory. However, in the present case, there is no initial mass scale, since the fluctuating particles, gluons and ghosts, are massless. In fact, it turns out to be impossible to establish a unique derivative expansion for the (inverse) gluon propagator by a simple counting of derivatives; this is because a typical expansion generates terms $\sim \frac{1}{|\partial_i A_0|}$, acting like a mass scale for the higher derivative terms. Therefore, we propose a different expansion scheme that is guided by the residual quasi-abelian gauge symmetry, which still holds for the background field.

The model is further specified by assuming that there are no magnetic field components in the rest frame of the heat bath; the latter is characterized by its 4-velocity vector u^μ . Therefore, there are only two independent (quasi-abelian) gauge invariants:

$$\begin{aligned} \mathbf{E}^2 &= \frac{1}{2} F_{\mu\nu} F_{\mu\nu} \equiv F_{\mu\alpha} u_\alpha F_{\mu\beta} u_\beta \\ \bar{A}_u &:= \frac{1}{\beta} \int_0^\beta d\tau A_u(x^\mu + \tau u^\mu), \quad A_u := A_\mu u_\mu. \end{aligned} \quad (3)$$

Here we work in Euclidean finite-temperature space $R^{d-1} \times S^1$, and $F_{\mu\nu}$ denotes the quasi-abelian field strength of the background field. In the heat-bath rest frame, we simply have $u^\mu = (1, \mathbf{0})$, so that $A_u \equiv A_0$. The quantity \bar{A}_u is invariant under quasi-abelian gauge transformations [12], since these transformations are restricted to be periodic in the compactified time direction. (For the complete gauge group, \bar{A}_u can be modified by a gauge transformation that differs at $x_0 = 0$ and $x_0 = \beta$ by a center element, e.g., $\bar{A}_u \rightarrow \frac{2\pi T}{g} - \bar{A}_u$ for $SU(2)$ modulo Weyl transformations.)

If we now perform a derivative expansion in the electric field \mathbf{E} , we will obtain an effective action of the form $\Gamma = f(\bar{A}_u) + g(\bar{A}_u) E^2 + \mathcal{O}(E^4, E\partial^2 E)$, ($E \equiv |\mathbf{E}|$, f, g to be determined) for reasons of gauge invariance. The indicated higher-order terms are at least of fourth order in ∂A and will be omitted in the following.

Now, the crucial observation is that there exists a unique choice of gauge for the background field that (i) satisfies the Polyakov gauge condition $\partial_0 A_0 = 0$ in order to ensure the correspondence between A_0 and L , and (ii) establishes a one-to-one correspondence between \mathbf{E} and A_0 , so that an expansion in \mathbf{E} can be rated as a derivative expansion in A_0

¹The spatially varying A_0 field giving rise to an electric field appears to conflict with the assumption of thermal equilibrium, which is inherent in the Matsubara formalism used below; this is because electric fields tend to separate (fundamental) color charges, moving the system away from equilibrium. However, we adopt the viewpoint that the here-considered vacuum model characterizes only a few features of the true vacuum; the latter actually includes quark and magnetic gluon condensates (and higher cumulants) which altogether are in equilibrium. Beyond this, we expect the present approximation to hold for sufficiently weak electric fields, keeping the system close to equilibrium. Therefore, the expansion in the electric field performed below is consistent with (almost) thermal equilibrium.

(from now on, we work in the heat-bath rest frame where $A_0 \equiv A_u$):

$$A_0(x) = a_0 - (x - x')_i E_i \quad \Leftrightarrow \quad E_i = -\partial_i A_0(x), \quad (4)$$

where a_0 and E_i are considered as constant, and x' is an arbitrary constant vector which can be set equal to zero. This gauge can be viewed as a combination of Polyakov and Schwinger-Fock gauge; the background field considered here lies exactly where the gauge conditions overlap. We remark that this is no longer true for higher derivative terms. The final task is to integrate out the thermal and quantum fluctuations in the background of the gauge field (4) and expand to second order in E_i .

At this point, it is useful to introduce the dimensionless temperature-rescaled variable

$$c := \frac{gA_0}{2\pi T}, \quad c \in [0, 1]. \quad (5)$$

The compactness of c arises from the fact that A_0 is a compact variable in finite-temperature Yang-Mills theories². Then, the resulting effective action can be represented as a derivative expansion in c :

$$\Gamma_{\text{eff}}^T[c] = \int d^d x \left(V(c, n^a) + W(c, n^a) \partial_i c \partial_i c \right), \quad (6)$$

where the potential V and the weight function W also depend on the color space unit vector n^a . Higher-order terms $\sim (\partial c)^4$ are neglected. The superscript T in Eq. (6) signals that the effective action strongly depends on the presence of a heat bath. Indeed, V as well as W vanish or reduce to simple constants as $T \rightarrow 0$; this is because the A_0 field (or \bar{A}_u in Eq. (3)) ceases to be an invariant at $T = 0$, since it can be gauged away completely when the time direction is noncompact.

Already at this stage, typical properties of the model become apparent. First, we observe that if $W(c, n^a) \geq 0$ fluctuations of the Polyakov loop are suppressed; then, the ground state is solely determined by the minimum (or minima) of $V(c, n^a)$, which we denote by c_V, n_V^a . This ground state then is (not) confining if it corresponds to a center (a-)symmetric state, implying $\langle L \rangle = 0$ ($\langle L \rangle \neq 0$).

Fluctuations of the Polyakov loop can only be preferred if $W(c, n^a)$ becomes negative for certain values of c and n^a , which we denote by c_W and n_W^a . Whether or not these fluctuations lead to a confining phase again depends on the question of whether or not the minimum of $W(c, n^a)$ corresponds to a center-symmetric state. Moreover, it depends on the question of whether these fluctuations are strong enough to compensate for the influence of $V(c, n^a)$. Here we arrive at a main problem of the model: if $W(c_W, n_W^a) < 0$, then the action (6) is not bounded from below. In other words, arbitrarily strong fluctuations of c around c_W will lower the action without any bound. Of course, it is reasonable to assume

²This can be inferred from a Hamiltonian quantization starting from the Weyl gauge $A_0 = 0$ and generating an A_0 field by a time-dependent gauge transformation. This observation will furthermore become obvious when studying the background field dependence of the gluon propagator (cf. Eq. (12)).

(which we do in the following) that higher derivative terms $(\partial c)^4$ or $(\partial^2 c)^2$ will establish such a lower bound, so that the strength of the fluctuations is dynamically controlled.

Nevertheless, one drawback remains: we cannot make any statement about the nature of a possible phase transition. For this, we would have to know everything about the dynamical increase of $\partial_i c \partial_i c$ when $W(c_W, n_W^a)$ becomes negative for certain values of temperature. Since this is beyond the capacities of our model, we shall always assume in the following that the system is dominated by the weight function W and thus by fluctuations of the Polyakov loop whenever W becomes negative.

Let us finally perform a dimensional analysis of the model. For simplicity, let us start with $d = 4$. With regard to Eq. (6), the potential has mass dimension 4, while the weight function has mass dimension 2. Due to the compactness of A_0 as reflected by Eq. (5), the only mass scale which is *a priori* present is given by the temperature T . Hence, if V scaled with T^4 and W with T^2 , say $V(c, n^a) = T^4 v(c, n^a)$ and $W(c, n^a) = T^2 w(c, n^a)$, where v, w are independent of T , then we would never encounter a phase transition in our model; this is because increasing or lowering the temperature could never turn W from positive to negative values or vice versa.

At this stage, one may speculate that, since scale invariance is broken in Yang-Mills theories, the phenomenon of dimensional transmutation introduces another scale μ (e.g., the scale at which the renormalized coupling is defined). Then, the dimensionless function w can also depend on T/μ . However, this is far from self-evident, since the breaking of scale invariance is induced by UV effects. But the functions V and W in the effective action Γ_{eff}^T arise at finite temperature only and thus are a product of infrared physics. In particular, there are no UV divergences in the finite-temperature contributions to Γ_{eff} which require another scale during a regularization procedure. Hence, one is tempted to conclude that the naive scaling argument given above is correct.

Nevertheless, the naive scaling breaks down, as we shall see in the next section; but this time, an additional scale is introduced by the properties of the theory in the infrared. As is well known, finite-temperature field theories can develop a more singular infrared behavior than their zero-temperature counterparts. Indeed, while the effective action at zero temperature and even the effective action for thermalized purely magnetic background fields do not suffer from infrared divergences, the case considered here involving thermalized electric fields exhibits such singularities, which must be handled carefully. The massless gluon does not provide for a natural infrared cutoff which could control the low-momentum behavior of the theory.

To conclude, the $d = 4$ model is in principle capable of describing a phase transition, because the finite-temperature infrared divergences require an additional scale which introduces distinct high- and low-temperature domains. Of course, there are various ways to deal with the infrared singularities; and as we will demonstrate below, they can arise from different physical motivations, leading to different physical results. Two possibilities are proposed in the present work. In the first and more natural one, we regularize the infrared divergences in the same technical way as the ultraviolet ones, so that in toto there is only one more scale than in the classical theory, which we identify with the defining scale of the coupling constant. As a consequence and a consistency check, the running of the

coupling with temperature coincides with the running of the coupling with field strength or momenta – they are characterized by the same β -function. In the second possibility, we study by way of example a regularization of the infrared divergences by an effective gluon mass m_{eff} which we insert by hand, assuming that such an additional scale may be generated dynamically in the full theory. The latter version of the theory exhibits the desired properties of two phases separated by a deconfinement phase transition, while the former does not.

At $d > 4$ the situation is somewhat different and simpler, since here the coupling constant g is dimensionful, so that two scales are present already at the classical level. Moreover, no additional scale will be introduced at the quantum level, because the theory is infrared finite for $d > 4$.

3 Calculation of the Effective Action

Starting from the standard formulation of Yang-Mills theories via the functional integral in Euclidean space with compactified time dimension, we employ the background field method [11] to fix the gauge for the fluctuating gluon fields, but thereby maintain gauge invariance for the background field. We arrive at the one-loop approximation by neglecting cubic and quartic terms in the fluctuating fields. The remaining two integrals over the gluonic and ghost fluctuations are Gaussian and lead to functional determinants upon integration; the one-loop effective action depending on the background field then reads

$$\Gamma_{\text{eff}}^1[A] = \frac{1}{2} \text{Tr}_{xcL} \ln \Delta^{\text{YM}}[A]^{-1} - \text{Tr}_{xc} \ln \Delta^{\text{FP}}[A]^{-1}, \quad (7)$$

where $\Delta^{\text{YM}}[A]^{-1}$ denotes the inverse gluon propagator, and $\Delta^{\text{FP}}[A]^{-1}$ the inverse ghost propagator, i.e., the Faddeev-Popov operator. The traces run over coordinate (x), color (c), and Lorentz (L) labels. Introducing the abbreviations $D^2 := D_\mu D_\mu$ and $(DD)_{\mu\nu} := D_\mu D_\nu$, where the covariant derivative is defined by $D_\mu := \partial_\mu - ig\mathbf{A}_\mu$, and suppressing the indices, the explicit representations of the propagators read

$$\Delta_{\text{E}}^{\text{YM}}[A]^{-1} = - \left[D^2 - 2ig F + \left(\frac{1}{\alpha} - 1 \right) DD \right], \quad (8)$$

$$\Delta_{\text{E}}^{\text{FP}}[A]^{-1} = -D^2, \quad (9)$$

In the following, we will work in the Feynman gauge, $\alpha = 1$, which simplifies the calculations considerably³. For the evaluation of Eq. (7), the spectrum of the inverse propagators is required. In color space, diagonalization can be achieved by introducing the eigenvalues ν_l of the matrix $n^a(T^a)^{bc}$, $l = 1, \dots, N_c^2 - 1$. The basic building block of the operators in

³For covariant constant background fields, the effective action is known to be independent of the gauge parameter [13]. In the present approximation, the gauge field (4) is covariant constant, so that we are allowed to set $\alpha = 1$; this would no longer remain true if we were interested in the higher-derivative terms.

Eqs. (8) and (9) is the covariant Laplacian, which upon insertion of the background field (4) yields (we set $x' = 0$)

$$(-iD[A])^2 = (-i\partial_i)^2 + (-iD_0[a_0])^2 + 2g\nu_l E_i (-iD_0[a_0])x_i + (g\nu_l)^2 E_i E_j x_i x_j, \quad (10)$$

where $-iD_0[a_0] = -i\partial_0 - g\nu_l a_0$, and the roman indices run over the $d-1$ spatial components. The operator is obviously of harmonic oscillator type and can be diagonalized by a rigid rotation of the spatial part of the coordinate system. A prominent eigenvector is given by the direction of the electric field E_i , which we may choose to point along the 1st direction of the new system. We finally obtain

$$(-iD[A])^2 = (-i\partial_2)^2 + \cdots + (-i\partial_{d-1})^2 + (g\nu_l E x_1 + (-iD_0))^2 + (-i\partial_1)^2, \quad (11)$$

where $E = \sqrt{E_i E_i}$. Up to now, we have achieved a partial diagonalization of the operators of Eq. (8) and (9). While the Faddeev-Popov operator coincides with the Laplacian (11), the inverse gluon propagator receives additional contributions from the gluon-spin coupling to the electric field $\sim -igF_{\mu\nu}$ which can easily be diagonalized. Performing a Fourier transformation for the $d-2$ unaffected components, $-i\partial_2, \dots, -i\partial_{d-1} \rightarrow p_2, \dots, p_{d-1}$, as well as for the time derivative, $-i\partial_0 \rightarrow \omega_n$, $\omega_n = 2\pi Tn$, $n \in \mathbb{Z}$ (Matsubara frequencies), we may write the inverse gluon propagator in the form

$$\Delta^{\text{YM}}[A]^{-1} = p_2^2 + \cdots + p_{d-1}^2 + (g\nu_l E x_1 + (\Pi_0))^2 + (-i\partial_1)^2 + 2\lambda g\nu_l E, \quad (12)$$

where $\Pi_0 = \omega_n - g\nu_l a_0$ ⁴. The number λ labels the different eigenvalues in Lorentz space arising from the above-mentioned gluon spin coupling with $\lambda = 1, -1, 0$; here, $\lambda = 1, -1$ appears only once, whereas $\lambda = 0$ occurs with multiplicity $d-2$, corresponding to the spatial directions which are unaffected by the electric field. Incidentally, the Faddeev-Popov operator is identical to Eq. (12) with $\lambda = 0$ and multiplicity 1. Taking the prefactors and signs of the two traces in Eq. (7) into account, the Faddeev-Popov operator cancels exactly against two Lorentz eigenvalues of the spectrum of $\Delta^{\text{YM}}[A]^{-1}$ with $\lambda = 0$, removing the spurious gauge degrees of freedom, so that only the physical, transverse part of the inverse gluon propagator remains,

$$\Delta_{\perp}^{\text{YM}}[A]^{-1} = p_2^2 + \cdots + p_{d-1}^2 + (e_l x_1 + (\Pi_0[a_l]))^2 + (-i\partial_1)^2 + 2\lambda e_l, \quad (13)$$

where $\lambda = 0$ now occurs with multiplicity $d-4$. For reasons of brevity, we introduced the short forms

$$e_l := |g\nu_l E|, \quad a_l := |g\nu_l a_0| \quad (14)$$

in Eq. (13); the use of the moduli in Eq. (14) is justified by the observation that, when tracing over a function of the inverse propagators, the result will not be sensitive to the signs of $g\nu_l E$ and $g\nu_l a_0$.

⁴The compactness of A_0 or a_0 becomes obvious here; e.g., for $\text{SU}(2)$, where $\nu_l = -1, 0, 1$, a shift of a_0 by an integer multiple of $(2\pi T)/g$ can be compensated for by a shift of the Matsubara label n .

The remaining problem of diagonalizing the 0-1 subspace at first sight resembles the problem of finding the spectrum of a relativistic particle in a constant magnetic field. There, one finds the eigenvalues (Landau levels) by shifting the x_1 coordinate by $x_1 \rightarrow x_1 - \frac{(-i\Pi_0)}{e_l}$ in order to arrive at a perfect harmonic oscillator. Here, the situation is not so simple, because the a_0 field as well as the temperature dependence would drop out of the operator completely. In other words, such a shift is not in agreement with the periodic boundary conditions in time direction.

Hence, the usual harmonic oscillator techniques arrive at their limits, and we have to find a different method that does not rely on the explicit knowledge of the spectra as is necessary for, e.g., ζ -function methods. We choose Schwinger's proper-time technique, which provides for a more direct handling of the propagators. In terms of the transverse gluon propagator, the effective action reads in proper-time representation

$$\begin{aligned}\Gamma_{\text{eff}}^1[A] &= \frac{1}{2} \text{Tr}_{\text{cL}} \ln \Delta_{\perp}^{\text{YM}}[A]^{-1} = -\frac{1}{2} \text{tr}_{\text{cL}} \int_0^{\infty} \frac{ds}{s} \langle x | e^{-s\Delta_{\perp}^{\text{YM}}[A]^{-1}} | x \rangle \\ &\equiv -\frac{1}{2} \text{tr}_{\text{cL}} \int_0^{\infty} \frac{ds}{s} \Omega \not\!\!\!\!\!\int \frac{d^d p}{(2\pi)^d} e^{-sM(p,\lambda,l;s)},\end{aligned}\tag{15}$$

where Ω denotes the spacetime volume of $\mathbb{R}^{d-1} \times S^1$, s is the proper time, and the trace over the continuous part of the spectrum is taken in momentum space. The color trace runs over l , which labels the color space eigenvalues, whereas the Lorentz trace runs over λ with its associated multiplicities. The function M is defined via the Fourier representation of the proper-time transition amplitude

$$\langle x | e^{-s\Delta_{\perp}^{\text{YM}}[A]^{-1}} | x' \rangle = \not\!\!\!\!\!\int \frac{d^d p}{(2\pi)^d} e^{ip(x-x')} e^{-sM(p,\lambda,l;s)},\tag{16}$$

which can be determined by the differential equation

$$\mathbb{1} = \Delta_{\perp}^{\text{YM}}[A]^{-1} \Delta_{\perp}^{\text{YM}}[A].\tag{17}$$

When evaluated, for example, in momentum space, Eq. (17) is solved by

$$\Delta_{\perp}^{\text{YM}}[A](p, \lambda, l) = \int_0^{\infty} ds e^{-sM(p,\lambda,l;s)},\tag{18}$$

where M is given by

$$\begin{aligned}M(p, \lambda, l; s) &= p_2^2 + \cdots + p_{d-1}^2 + \frac{\tanh 2e_l s}{2e_l s} (p_1 + q)^2 + \frac{\tanh e_l s}{e_l s} (\omega_n - a_l)^2 \\ &\quad + \frac{1}{2s} \ln \cosh 2e_l s + 2e_l \lambda.\end{aligned}\tag{19}$$

Here, q denotes some function of e_l and s which becomes irrelevant when shifting the p_1 integration in Eq. (15). Upon insertion of Eq. (19) into Eq. (15), the Gaussian momentum integration and the sum over λ can easily be performed; the sum over Matsubara frequencies can be reorganized by a simple Poisson resummation,⁵ and we arrive at

$$\begin{aligned} \Gamma_{\text{eff}}^1[A] = & -\frac{\Omega}{2} \text{tr}_c \frac{1}{(4\pi)^{d/2}} \int_0^\infty \frac{ds}{s^{d/2}} e_l \left(4 \sinh e_l s + \frac{d-2}{\sinh e_l s} \right) \\ & \times \left[1 + 2 \sum_{n=1}^\infty \exp \left(-\frac{n^2}{4T^2} e_l \coth e_l s \right) \cos \frac{a_l}{T} n \right]. \end{aligned} \quad (20)$$

Here, we have separated the zero-temperature part, corresponding to the first line times the “1” of the second line, from the finite-temperature contributions, corresponding to the first line read together with the n sum.

3.1 Effective Action at Zero Temperature

Let us first study the temperature-independent part of the effective action Eq. (20) with particular emphasis on its renormalization:

$$\Gamma_{\text{eff}}^{1T=0}[A] = -\frac{\Omega}{2} \text{tr}_c \frac{1}{(4\pi)^{d/2}} \int_0^\infty \frac{ds}{s^{d/2}} e_l \left(4 \sinh e_l s + \frac{d-2}{\sinh e_l s} \right). \quad (21)$$

On the one hand, the proper-time integral is divergent at the upper bound, $s \rightarrow \infty$, owing to the first term $\sim \sinh e_l s$. Since large values of s correspond to the infrared regime, this divergence is not related to the standard renormalization of bare parameters, which is a UV effect. In fact, this divergence is analogous to the Nielsen-Olesen unstable mode [9] of the Savvidy vacuum⁶; one can give a meaning to this essential singularity by rotating the contour of the integral over the \sinh term into the lower complex plane, $-is \rightarrow s$. The effective action then picks up an imaginary part that characterizes the instability of the constant electric background field considered here.

On the other hand, the proper-time integral is also divergent at the lower bound, corresponding to the ultraviolet. The leading singularity is of the order $s^{-d/2}$, so that m subtractions are required for $d = 2m$ or $d = 2m + 1$. The leading singularity which is field independent can easily be removed by demanding that $\Gamma_{\text{eff}}^{T=0}[A = 0] = 0$ (first renormalization condition). The next-to-leading singularity proportional to $e_l^2 \sim E^2$ is removed by the second renormalization condition $(\partial \mathcal{L}_{\text{eff}} / \partial E^2)|_{E \rightarrow 0} = 1/2$, where $\Gamma_{\text{eff}} = \int \mathcal{L}_{\text{eff}}$; this ensures that the classical Lagrangian is recovered when all nonlinear interactions

⁵For technical details, see, e.g., [12, 14].

⁶At $T = 0$, the present situation involving an external *electric* field is identical to the *magnetic* Savvidy vacuum owing to the Euclidean $O(4)$ symmetry.

are switched off, and corresponds to a field-strength and charge renormalization

$$\mathcal{L}_{\text{cl}}^{\text{R}} \equiv \frac{1}{2} E_{\text{R}}^2 = \frac{1}{2} Z_3^{-1} E^2, \quad (22)$$

where E_{R} denotes the renormalized field, and Z_3 is the wave function renormalization constant. The latter can be read off from Eq. (21) by isolating the singularity $\sim E^2$,

$$Z_3^{-1} = 1 - \frac{26-d}{6(4\pi)^{d/2}} N_{\text{c}} \bar{g}^2 \int_{\mu^2/\Lambda^2}^{\infty} \frac{ds}{s^{d/2-1}}, \quad (23)$$

where we have used an explicit cutoff Λ , employed $\text{tr}_{\text{c}} |\nu_l|^2 = \sum_{l=1}^{N_{\text{c}}^2-1} |\nu_l|^2 = N_{\text{c}}$, and have introduced the dimensionless coupling $\bar{g}^2 = g^2 \mu^{d-4}$ with the aid of a reference scale μ (at which g is defined). To one-loop order, the β function can be read off from the coefficient of the UV divergence of Z_3^{-1} :

$$\beta_{\bar{g}^2} \equiv \partial_t \bar{g}^2 = (d-4) \bar{g}^2 - b_0^d \bar{g}^4, \quad (24)$$

where

$$\begin{aligned} b_0 &= \frac{11}{3} \frac{N_{\text{c}}}{8\pi^2}, \quad \text{for } d=4, \\ b_0^d &= \frac{(26-d)}{3(d-4)} \frac{N_{\text{c}}}{(4\pi)^{d/2}}, \quad \text{for } d>4, \end{aligned} \quad (25)$$

and t denotes the “renormalization group time” $\ln \mu/\Lambda$. Here we have rediscovered the standard well-known one-loop results, including the remarkable observation that the β function for the dimensionful coupling $g^2 = \bar{g}^2 \mu^{4-d}$ vanishes precisely in the critical string dimension $d=26$ [15].

Note that in $4 < d < 26$, the β function develops a UV-stable fixed point:

$$\bar{g}_*^2 = \frac{d-4}{b_0^d} = \frac{3(d-4)^2}{26-d} \frac{(4\pi)^{d/2}}{N_{\text{c}}}. \quad (26)$$

Of course, this fixed point lies in the perturbative domain ($\bar{g}_*^2/4\pi \ll 1$) only for very large N_{c} .

As an alternative to these considerations of renormalization, the integral in Eq. (21) can be treated more directly with an appropriate regularization prescription. Let us briefly sketch a proper-time variant of dimensional regularization for later use in the case $d=4$. Shifting the singularities at $s \rightarrow 0$ by ϵ and introducing a mass scale μ , Eq. (21) can be written as

$$\mathcal{L}_{\text{eff}}^{1T=0} = -\frac{1}{8\pi^2} \text{tr}_{\text{c}} \mu^{2\epsilon} \left[\int_0^{-i\infty} \frac{ds}{s^{2-\epsilon}} e_l \sinh e_l s + \frac{d-2}{4} \int_0^{\infty} \frac{ds}{s^{2-\epsilon}} \frac{e_l}{\sinh e_l s} \right]. \quad (27)$$

These integrals can be evaluated [16], and the result for the one-loop contribution to the zero-temperature effective Lagrangian in $d = 4$ reads

$$\mathcal{L}_{\text{eff}}^{1T=0} = -\frac{1}{8\pi^2} \text{tr}_c e_l^2 \left[\frac{11}{12\epsilon} - \frac{11}{12} \ln \frac{e_l}{\mu^2} + \text{const.} + \text{imag. parts} + \mathcal{O}(\epsilon) \right]. \quad (28)$$

The appearance of the simple pole in ϵ implies a charge and field strength renormalization as outlined above. To be precise, in the background field formulation, the coupling runs with the scale set by the strength of the external *field*: $g^2 = g^2(gE/\mu^2)$; this is analogous to the *momentum* dependence of the coupling in the standard formulation. Including the correctly (re-)normalized classical term, the total effective Lagrangian to one loop can then be written as

$$\mathcal{L}_{\text{eff}}^{T=0}(gE) = \mathcal{L}_{\text{cl}} + \mathcal{L}_{\text{eff}}^{1T=0} = \frac{1}{8} b_0 (gE)^2 \ln \frac{(gE)^2}{e\kappa^2} + \text{imag. parts}, \quad \kappa^2 = \mu^4 e^{-\frac{4}{b_0 g^2} - 1}, \quad (29)$$

where we have introduced the renormalization group invariant quantity κ corresponding to the minimum of $\mathcal{L}_{\text{eff}}^{T=0}(gE)$, and b_0 is given by the first line of Eq. (25).

Concerning the imaginary parts, the following comment should be made: within the Savvidy model, the imaginary parts indicate the instability of the constant-field vacuum configuration signaling the final failure of the model. In the present case, they are just an artefact of truncating the derivative expansion of the effective action at second order; this truncation is formally equivalent to the constant-field approximation. Upon an inclusion of non-constant terms which would affect only higher-derivative contributions, we expect the imaginary part to vanish; this is because the unstable modes are then cut off by the length scale of variation of the fields.

For the regularization/renormalization program of $\mathcal{L}_{\text{eff}}^{1T=0}$ in $d \geq 6$, more subtractions than in $d = 4$ are needed; since these theories are nonrenormalizable in the common sense, these subtractions correspond to counter-terms of field operators of higher mass dimension. To be precise, $d > 4$ Yang-Mills theories can only be defined with a cutoff (with physical relevance); therefore, some cutoff procedure is implicitly understood. Nevertheless, the precise form of the cutoff procedure only affects higher-order operators which are of no relevance for the present model. Moreover, it is perfectly legitimate to study $d > 4$ quantum Yang-Mills theories in the sense of effective theories valid below a certain cutoff scale.

3.2 Effective Action at Finite Temperature

Equipped with these preliminaries, we now turn to the more interesting finite-temperature part of Eq. (20):

$$\mathcal{L}_{\text{eff}}^{1T} = -\frac{1}{(4\pi)^{d/2}} \text{tr}_c \int_0^\infty \frac{ds}{s^{d/2}} e_l \left(4 \sinh e_l s + \frac{d-2}{\sinh e_l s} \right) \sum_{n=1}^\infty \exp \left(-\frac{n^2}{4T^2} e_l \coth e_l s \right) \cos \frac{a_l}{T} n. \quad (30)$$

For $s \rightarrow 0$, the integral remains completely finite, since the coth in the exponent develops a $1/s$ pole; i.e., there are no UV divergences in the thermal contribution to \mathcal{L}_{eff} , as is to be expected.

At the opposite end, $s \rightarrow \infty$, we again encounter the sinh divergence induced by the unstable mode. However, this is not the only infrared problem: an attempt at circumventing this problem by a rotation of the s contour as in the zero-temperature case would lead to a disastrous behavior of the n sum due to the poles of the coth on the imaginary axis. In fact, it is the interplay between the proper-time integral and the n sum that produces further infrared divergences (at least for $d=4$). It is well known in the literature that particles with Bose-Einstein statistics develop stronger infrared singularities at $T \neq 0$ than at zero temperature [17]. Unfortunately, the status of these finite-temperature singularities is far from being settled, contrary to the $T = 0$ case.

In the present paper, we shall investigate two different methods. The consequences of an explicit mass-like cutoff are discussed in Sec. 5. Here, we propose a more natural treatment by regularizing the thermal infrared divergences of Eq. (30) by the same method used to treat the UV divergences in Eq. (27) in the $T = 0$ case. Thereby, the same scale μ which serves to define the value of the coupling constant is introduced.

Taking these considerations into account, the Lagrangian is modified according to (substitution $\mu^2 s = u$)

$$\begin{aligned} \mathcal{L}_{\text{eff}}^{1T} = & -\frac{4}{(4\pi)^{d/2}} \text{tr}_c \mu^d \int_0^\infty \frac{du}{u^{d/2-\epsilon}} \left(\frac{e_l}{\mu^2} \sinh \frac{e_l}{\mu^2} u + \frac{d-2}{4} \frac{e_l/\mu^2}{\sinh \frac{e_l}{\mu^2} u} \right) \\ & \times \sum_{n=1}^\infty \exp \left(-\frac{n^2}{4} \frac{\mu^2}{T^2} \frac{e_l}{\mu^2} \coth \frac{e_l}{\mu^2} u \right) \cos \frac{a_l}{T} n. \end{aligned} \quad (31)$$

In the context of our approximation in terms of derivatives of A_0 , we need only the terms $\sim e_l^0$ and $\sim e_l^2$ of Eq. (31). Expanding in e_l/μ^2 and performing the s integral, we arrive at

$$\mathcal{L}_{\text{eff}}^{1T}|_0 = -\frac{(d-2)\Gamma(d/2)}{\pi^{d/2}} \sum_{l=1}^{N_c^2-1} \sum_{n=1}^\infty \frac{\cos \frac{a_l}{T} n}{n^d} T^d =: V(c, n^a), \quad (32)$$

$$\begin{aligned} \mathcal{L}_{\text{eff}}^{1T}|_{e_l^2} = & -\frac{1}{6\pi^{d/2}} \sum_{l=1}^{N_c^2-1} \left(\frac{e_l}{\mu^2} \right)^2 \sum_{n=1}^\infty \frac{\cos \frac{a_l}{T} n}{n^d} \left(\frac{n^2 \mu^2}{4T^2} \right)^{2+\epsilon} \\ & \times \Gamma(d/2-2-\epsilon) [(26-d) - (d-2)(d-4-2\epsilon)] T^d, \end{aligned} \quad (33)$$

For the term $\sim e_l^0$ in the first line, the $\epsilon \rightarrow 0$ limit could safely be performed for $d \geq 0$; by construction, this term depends only on $a_l \sim a_0 \sim c$ (cf. Eq. (5)) and therefore corresponds to the potential $V(c, n^a)$ as introduced in Eq. (6).

The term $\sim e_l^2$ in the second line contributes to the function $W(c, n^a)$ (in addition to the classical term). It turns out that, for $d > 4$, the limit $\epsilon \rightarrow 0$ can be performed immediately without running into an ϵ pole. This means that, in these dimensions, the

thermally modified infrared behavior of the theory is under control. The order e_l^2 term of the one-loop effective action then reads

$$\mathcal{L}_{\text{eff}}^{1T}|_{e_l^2} = -\frac{\Gamma(d/2-2)}{96\pi^{d/2}} \sum_{l=0}^{N_c^2-1} \left(\frac{e_l}{T^2}\right)^2 \sum_{n=1}^{\infty} \frac{\cos \frac{a_l}{T} n}{n^{d-4}} [(26-d) - (d-2)(d-4)] T^d, \quad d > 4. \quad (34)$$

Obviously, the μ dependence has dropped out as a consequence of the well-behaved $\epsilon \rightarrow 0$ limit. Nevertheless, there is a second scale besides the temperature, which is given by the dimensionful coupling constant g in $d > 4$.

In $d = 4$, the situation is more involved, since Eq. (33) develops a simple pole in ϵ for $\epsilon \rightarrow 0$. In order to isolate the pole and the terms of order ϵ^0 which contain the physics, we first have to perform the n sum; this can be achieved with the aid of the polylogarithmic function (also Jonquières function)

$$\text{Li}(z, q) := \sum_{n=1}^{\infty} \frac{q^n}{n^z} \quad (35)$$

and its analytical continuation for arbitrary real values of z [18]. We finally find for Eq. (33) in $d = 4$:

$$\begin{aligned} \mathcal{L}_{\text{eff}}^{1T}|_{e_l^2} = & -\frac{1}{8\pi^2} \sum_{l=1}^{N_c^2-1} e_l^2 \left[\frac{11}{12\epsilon} - \frac{11}{12} \ln \frac{T^2}{\mu^2} + \frac{11}{6} \text{Li}'(0, e^{i\frac{a_l}{T}}) + \frac{11}{6} \text{Li}'(0, e^{-i\frac{a_l}{T}}) \right. \\ & \left. + \frac{1}{6} + \frac{11}{12} C - \frac{11}{12} \ln 4 \right], \quad d = 4, \end{aligned} \quad (36)$$

where the prime at Li denotes the derivative with respect to the first argument, and C is Euler's constant $C \simeq 0.577216$. Our first observation is that the ϵ pole in this thermal contribution is identical to the one for the zero-temperature Lagrangian in Eq. (28). Since the latter is responsible for the usual charge and field strength renormalization leading to a field-strength-dependent coupling $g^2 = g^2(gE/\mu^2)$, the present ϵ pole analogously suggests a running of the coupling with the scale set by the temperature: $g^2 = g^2(T^2/\mu^2)$. And because the residues of each pole are identical, the thermal running is governed by the same β function. This can be viewed as a consistency check of our treatment of the infrared singularities.

Furthermore, the terms $\sim \epsilon^0$ depend on the ratio T^2/μ^2 (even in the limit $a_l \rightarrow 0$). This implies that they cannot be normalized away as in the zero-temperature case, but lead to a thermal renormalization of the two-point function.

This is in perfect analogy to QED, where an equivalent modification of the two-point function appears with the prefactor (= Yang-Mills β function) replaced by the QED β function, and the role of μ is played by the natural scale of QED: the electron mass [19, 14].

In conclusion, it is the $\ln \frac{T^2}{\mu^2}$ term in Eq. (36) which leads to a breakdown of the naive scaling as outlined in Sec. 2 and allows for a separation of high- and low- temperature

regimes. This could in principle facilitate a description of a phase transition within the $d = 4$ model. However, as we shall find in the next section, the model does not make use of this option.

4 Analysis of the Effective Action

In the following analysis of the previously derived effective action for arbitrary d and N_c , for simplicity we confine ourselves to $N_c = 2$, which provides for a convenient study of all the essential features of the model. Then, the color space eigenvalues ν_l are simply given by

$$\nu_l = -1, 0, 1, \quad \text{for } \text{SU}(2). \quad (37)$$

The results given above can be summarized in the effective Lagrangian (cf. Eqs. (5) and (6)):

$$\mathcal{L}_{\text{eff}}^T[c] = V(c) + W(c) \partial_i c \partial_i c, \quad (38)$$

where we have used the relations (cf. Eq. also (4))

$$c = \frac{ga_0}{2\pi T}, \quad \text{and} \quad \partial_i c = \frac{-gE_i}{2\pi T}, \quad c \in [0, 1]. \quad (39)$$

The convenient dimensionless quantity c is now considered as the dynamical variable of the effective theory; for $\text{SU}(2)$, the center symmetric point is given by $c = 1/2$, since center symmetry relates c with $1 - c$. If the vacuum state is characterized by $c = 1/2$, our model is confining, whereas a vacuum state different from $c = 1/2$ characterizes the deconfinement phase.

4.1 Four Dimensions $d = 4$

Beginning with the most relevant case of four spacetime dimensions, the potential can be read off from Eq. (32). Performing the n sum leads to a Bernoulli polynomial,

$$V(c) = -\frac{3\pi^2}{45} T^4 + \frac{4\pi^2}{3} T^4 c^2(1 - c)^2, \quad (40)$$

in agreement with [6]. While the first term is simply the free energy of $N_c^2 - 1 = 3$ free gluons, the second models the shape of the potential revealing a maximum at $c = 1/2$ and minima at $c = 0, 1$ and thereby characterizing the deconfinement phase (see Fig. 1(a)). However, even if the potential had displayed a minimum at $c = 1/2$, it would have been of no use, since the potential by itself would remain confining for arbitrarily high temperatures. There would be no comparative scale separating two different phases. A Polyakov loop potential depending on c and T only can never model the deconfinement phase transition of Yang-Mills theories!

The weight function $W(c)$ can be read off from Eq. (36) in combination with the classical Lagrangian $\mathcal{L}_{\text{cl}} = E^2/2 = \frac{2\pi^2 T^2}{g^2(\mu)} \partial_i c \partial_i c$:

$$\begin{aligned} W(c) &= 2\pi^2 T^2 \left\{ \frac{1}{g^2(\mu)} - b_0 \left[-\ln \frac{T}{\mu} + \frac{C}{2} + \frac{1}{11} - \ln 2 + \text{Li}'(0, e^{2\pi i c}) + \text{Li}'(0, e^{-2\pi i c}) \right] \right\} \\ &= 2\pi^2 T^2 b_0 \left[\ln \frac{T}{\sqrt{\kappa}} - \frac{1}{4} - \frac{1}{11} - \frac{C}{2} + \ln 2 - \text{Li}'(0, e^{2\pi i c}) - \text{Li}'(0, e^{-2\pi i c}) \right], \end{aligned} \quad (41)$$

where b_0 denotes the β function coefficient given in the first line of Eq. (25) (for $N_c = 2$). In the second line, we have expressed the running coupling and the scale μ by the renormalization group invariant κ defined in Eq. (29), so that $W(c)$ is itself renormalization group invariant! In fact, lowering the temperature can turn the weight function negative for any value of c so that fluctuations of the Polyakov loop are energetically preferred for low T . However, the confining value $c = 1/2$ always represents a local maximum of the weight function $W(c)$, as is visible in Fig. 1(b). For $c \rightarrow 0, 1$, the weight function diverges to $-\infty$, but at $c = 1, 0$ it jumps to its absolute maxima. Analytically, one finds

$$W([c = 0, 1; c = 1/2]) = 2\pi^2 T^2 b_0 \left([\ln 4\pi; \ln \pi] - \frac{15}{44} - \frac{C}{2} + \ln \frac{T}{\sqrt{\kappa}} \right). \quad (42)$$

To conclude, although our model indicates that fluctuations of the Polyakov loop become important at low temperatures, they do not fluctuate around the confining minimum, but energetically prefer a center asymmetric ground state for c . Hence, our model is not capable of finding a confinement phase⁷. Nevertheless, it should be stressed that the present treatment of the infrared modes is part of the definition of the model, although we have tried to formulate the present version as “universally” as possible. In fact, the regularization method considered here, which belongs to the standard class of regularization techniques, guarantees scheme-independent results. But it is also possible that the infrared modes are screened by a physical mechanism which involves another scale and thereby introduces “nonuniversal” information. Such a version of the model is discussed by way of example in Sec. 5.

⁷The discontinuous behavior of the weight function for $c \rightarrow 0, 1$ gives rise to speculations. Physically, such behavior is not acceptable (nor interpretable); rather, one may expect that some mechanism will lead to a wash-out of these singularities unveiling a smooth functional form of $W(c)$ for $c \in [0, 1]$ (although the origin of such a mechanism is still unclear to us). Probably, this will lead to a weight function of mexican-hat type with deconfining minima. However, with even more reservations, one might speculate upon the possibility of a smooth curve for $W(c)$ which directly interpolates between the extremal values at $c = 0, 1/2, 1$ given in Eq. (42) with a confining minimum at $c = 1/2$. Then, the model would exhibit a confining phase for small enough temperatures when $W(c)$ becomes negative for $c = 1/2$. The reason for mentioning such vague speculations is to demonstrate how possible predictions could in principle arise from the model: following the reasoning of Sec. 2, the temperature of the phase transition is then given by $W(c = 1/2)|_{T=T_{\text{cr}}} = 0$. From Eq. (42), we obtain: $T_{\text{cr}}/\sqrt{\kappa} \simeq 0.60$. Identifying κ with the string tension σ (as it is the case in the leading-log model [20]), our speculative estimate is in remarkably good agreement with the lattice value [21], $T_{\text{cr}}/\sqrt{\sigma} \simeq 0.69$.

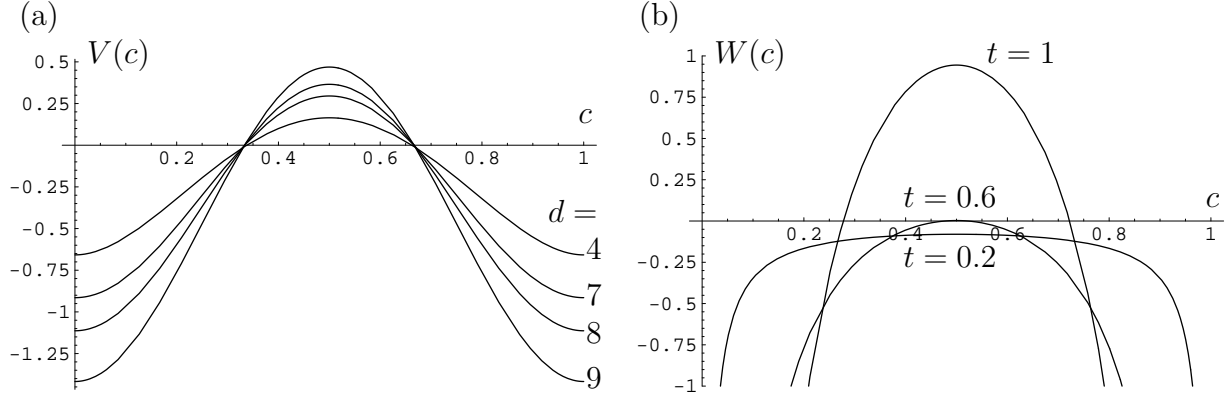


Figure 1: (a) SU(2)-Polyakov loop potential $V(c)$ in units of T^d for $d = 4, 7, 8, 9$ (cf. Eqs. (40) and (43)). The $d = 4, 8$ potentials correspond to Bernoulli polynomials B_4 and B_8 . (b) SU(2) weight function $W(c)$ in units of κ in $d = 4$ for different values of the temperature $t := T/\sqrt{\kappa} = 0.2, 0.6, 1$ (cf. Eq. (41)). The disconnected absolute maxima at $W(c = 0, 1)$ are not depicted.

4.2 Beyond Four Dimensions $d > 4$

In spacetime dimensions larger than four, the situation simplifies owing to the absence of infrared problems. The Polyakov loop potential is again given by Eq. (32), which, for $N_c = 2$, reads

$$V(c) = -\frac{(d-2)\Gamma(d/2)\zeta(d)}{\pi^{d/2}} T^d - \frac{2}{\pi^{d/2}}(d-2)\Gamma(d/2) \sum_{n=1}^{\infty} \frac{\cos 2\pi cn}{n^d} T^d, \quad (43)$$

where $\zeta(d)$ denotes Riemann's ζ function. Equation (43) is in perfect agreement with [22], where it is demonstrated that a representation of $V(c)$ in terms of Bernoulli polynomials of d th degree exists in $d = 2, 4, 6, 8, \dots$. We could as well choose a representation in terms of polylogarithmic functions which interpolate smoothly between the Bernoulli polynomials. In toto, the qualitative behavior of $V(c)$ does not change significantly for different d : $V(c = 1/2)$ is always a (deconfining) maximum (cf. Fig. 1(a)).

The situation is different for the weight function $W(c)$: in terms of the dimensionless coupling $\bar{g}^2 = \mu^{d-4}g^2$ and polylogarithmic functions, the contributions from Eq. (34) together with the classical Lagrangian can be represented as

$$W(c) = 2\pi^2 \frac{T^2}{\mu^2} \mu^{d-2} \left\{ \frac{1}{\bar{g}^2} - \frac{T^{d-4}}{\mu^{d-4}} \frac{\Gamma(d/2-2)}{48\pi^{d/2}} [(26-d) - (d-2)(d-4)] \right. \\ \left. \times \left[\text{Li}(d-4, e^{2\pi ic}) + \text{Li}(d-4, e^{-2\pi ic}) \right] \right\}. \quad (44)$$

On the one hand, we again encounter the combination of polylogarithmic functions that interpolate between the Bernoulli polynomials of $(d-4)$ th degree for $d = 6, 8, \dots$, essentially maintaining their typical shape. On the other hand, there is an important sign

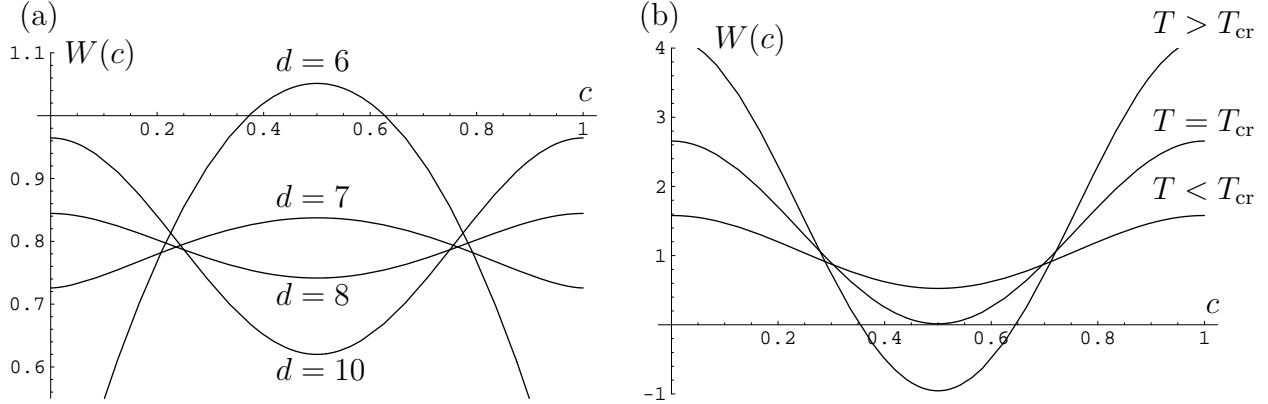


Figure 2: (a) SU(2) weight function $W(c)$ in units of μ for $d = 6, 7, 8, 10$ and fixed T and \bar{g} (cf. Eq. (44)). Above $d = d_{\text{cr}} \simeq 7.42$, $c = 1/2$ represents the minimum of $W(c)$. (b) The same weight function is now plotted for fixed \bar{g} and $d = 10 > d_{\text{cr}}$ for various temperature values close to T_{cr} .

change owing to the factor $(26-d) - (d-2)(d-4)$ at the “critical dimension”

$$d_{\text{cr}} = \frac{1}{2}(5 + \sqrt{97}) \simeq 7.42. \quad (45)$$

For $d < d_{\text{cr}}$, $W(c)$ has a maximum at $c = 1/2$, implying that there is no confining phase in these dimensions. But for $d > d_{\text{cr}}$, the weight function exhibits an absolute minimum at the center symmetric value $c = 1/2$ (see Fig. 2(a)). As a consequence, $W(c)$ can become negative at $c = 1/2$ for *increasing* temperature, as is depicted in Fig. 2(b). This is in agreement with the fact that the dimensionless coupling grows large in the *high*-momentum limit with a UV-stable fixed point given by Eq. (26). Therefore, the model, somewhat counter-intuitively, describes a system with two different phases, a deconfined phase at low temperature and a confining strong-coupling phase at high temperature. In terms of the dimensionful coupling constant, the critical temperature where $W(c = 1/2)|_{T=T_{\text{cr}}} = 0$ is given by

$$g^2 T_{\text{cr}}^{d-4} = \frac{24\pi^{d/2}}{\Gamma(d/2 - 2)\zeta(d-4)} \frac{2^{d-5}}{(2^{d-5} - 1)[(d-2)(d-4) - (26-d)]}, \quad d > d_{\text{cr}}. \quad (46)$$

Because of the strong increase of the Γ and ζ function in the denominator, the left-hand side rapidly falls off for increasing d . Typical values are $g^2 T_{\text{cr}}^{d-4} \simeq 411.4, 12.0, 0.036$ for $d = 8, 16, 26$. Therefore, the deconfined phase vanishes in the formal limit $d \rightarrow \infty$.

Incidentally, it is interesting to observe that the discontinuities of the weight function $W(c)$ for $c \rightarrow 0, 1$ vanish for $d > 5$; there, $W(c)$ runs continuously to a finite extremal value for $c \rightarrow 0, 1$. Between four and five dimensions, the discontinuity persists and $W(c = 0, 1)$ increases for increasing d , finally approaching plus infinity at $d \rightarrow 5^-$.

5 Additional Infrared Scales in $d = 4$

The preceding section revealed that the $d = 4$ model required additional instructions on how to treat the singular infrared modes. Although we rate the procedure established above as the most general one of a “universal” character, we shall now suggest another method, involving an additional scale. In the following investigation, we exemplarily pick out one (physically motivated) possibility of regularizing the infrared modes, and study its consequences.

Let us assume that Yang-Mills theory dynamically generates a scale in the infrared which can be reformulated in terms of an effective mass⁸ m_{eff} for the transverse fluctuating gluons⁹. Although this scale may in itself depend on some parameters, we shall consider it to be constant within the limits of our investigation.

Adding the effective mass term to the inverse transverse gluon propagator, e.g., in Eq. (13), it appears in a standard way in the proper-time representation of the effective action; for example, the integrand of the thermal one-loop contribution in Eq. (30) is multiplied by $e^{-m_{\text{eff}}^2 s}$ which damps away the infrared singularities. Upon the substitution $u = m_{\text{eff}}^2 s$, we obtain

$$\begin{aligned} \mathcal{L}_{\text{eff}}^{1T} = & -\frac{m_{\text{eff}}^4}{4\pi^2} \text{tr}_c \int_0^\infty \frac{du}{u^2} e^{-u} \frac{e_l}{m_{\text{eff}}^2} \left(\sinh \frac{e_l}{m_{\text{eff}}^2} u + \frac{1}{2 \sinh \frac{e_l}{m_{\text{eff}}^2} u} \right) \\ & \times \sum_{n=1}^\infty \exp \left(-\frac{n^2}{4} \frac{m_{\text{eff}}^2}{T^2} \frac{e_l}{m_{\text{eff}}^2} \coth \frac{e_l}{m_{\text{eff}}^2} u \right) \cos \frac{a_l}{T} n. \end{aligned} \quad (47)$$

Expanding for small e_l/m_{eff}^2 in order to arrive at a consistent derivative expansion for A_0 , we find to order e_l^0 and e_l^2

$$\mathcal{L}_{\text{eff}}^{1T}|_0 = -\frac{1}{\pi^2} m_{\text{eff}}^2 \text{tr}_c \sum_{n=1}^\infty \frac{T^2}{n^2} K_2(m_{\text{eff}} n/T) \cos \frac{a_l}{T} n \equiv V(c, n^a, m_{\text{eff}}), \quad (48)$$

$$\begin{aligned} \mathcal{L}_{\text{eff}}^{1T}|_{e_l^2} = & -\frac{11}{24\pi^2} \text{tr}_c e_l^2 \sum_{n=1}^\infty K_0(m_{\text{eff}} n/T) \cos \frac{a_l}{T} n \\ & + \frac{1}{24\pi^2} \text{tr}_c e_l^2 \sum_{n=1}^\infty \left(\frac{m_{\text{eff}}}{T} n \right) K_1(m_{\text{eff}} n/T) \cos \frac{a_l}{T} n, \end{aligned} \quad (49)$$

where we have employed a representation of the modified Bessel function $K_\nu(x)$ [16]. Since we are interested in a possible formation of a confinement phase, let us study Eqs. (48)

⁸This mass should not be associated with a thermal gluon mass; the latter represents a collective excitation of the thermal plasma and is a typical feature of the high-temperature domain, being proportional to T . By contrast, the effective mass considered here shall particularly affect the low-temperature modes and be approximately constant in T .

⁹In this way, gauge invariance with respect to the background field is maintained.

and (49) in the low-temperature limit $T \ll m_{\text{eff}}$. Then it is sufficient to use the asymptotic form of the Bessel functions for large argument, $K_\nu(x) \rightarrow \sqrt{\pi/2x}e^{-x}$.

Confining ourselves to the simplest case $\text{SU}(2)$, we can deduce the form of the potential from Eq. (48):

$$V(c, m_{\text{eff}}) \simeq -\sqrt{\frac{2}{\pi^3}} T^4 \left(\frac{m_{\text{eff}}}{T}\right)^{3/2} e^{-m_{\text{eff}}/T} \left(\cos 2\pi c + \frac{1}{2}\right), \quad T \ll m_{\text{eff}}. \quad (50)$$

Again, we encounter a potential with a (deconfining) maximum at $c = 1/2$, so that the effective mass does not induce significant changes to the potential term.

Including the contribution from the classical Lagrangian, the weight function can be deduced from Eq. (49) in the same limit:

$$W(c, m_{\text{eff}}) = T^2 \left[\frac{2\pi^2}{g^2} - \frac{1}{3} \sqrt{\frac{\pi}{2}} \left(11 \sqrt{\frac{T}{m_{\text{eff}}}} - \sqrt{\frac{m_{\text{eff}}}{T}} \right) e^{-m_{\text{eff}}/T} \cos 2\pi c \right]. \quad (51)$$

We first observe that, since the m_{eff} dependent term is exponentially small for $T \ll m_{\text{eff}}$, a small coupling g^2 will always ensure that $W(c, m_{\text{eff}})$ is positive so that Polyakov loop fluctuations are suppressed and the system is in the deconfined phase. Therefore, the model predicts that confinement requires a strong coupling.

Indeed, if the coupling is (very) strong, we may neglect the first term in Eq. (51), and find that $W(c, m_{\text{eff}})$ develops a minimum at $c = 1/2$, if¹⁰

$$T < T_{\text{cr}}, \quad \frac{T_{\text{cr}}}{m_{\text{eff}}} = \frac{1}{11}, \quad \text{for } g^2 \gg 1. \quad (52)$$

The situation can be rephrased as follows: if $T < T_{\text{cr}}$, $c = 1/2$ is the absolute minimum of $W(c, m_{\text{eff}})$. But $W(c = 1/2, m_{\text{eff}})$ only becomes negative (thereby allowing for a confinement phase) if the coupling is sufficiently large, so that the first term of Eq. (51) can be neglected.

Therefore, our main conclusion of the present section is that a different treatment of the infrared modes changes the behavior of the model significantly! Although the present version of the model exhibits the desired features, it requires more input and thus is less meaningful: we need to specify the value of m_{eff} and the value of g^2 ; the latter is involved with another scale μ .

Let us end this section with the comment that the introduction of a masslike infrared cutoff as employed in Eq. (47) can also be used as an alternative regularization scheme for the infrared modes. This means that, giving up the meaning of m_{eff} as a physical scale, but treating it as an arbitrary infrared cutoff scale for Eq. (47), we may remove it after the calculation by taking the limit $m_{\text{eff}} \rightarrow 0$ in Eqs. (48) and (49). We exactly recover Eq. (32) (for $d = 4$), and, after analytical continuation, also Eq. (36) with the association $m_{\text{eff}} \sim \mu$. The same procedure in $d > 4$ dimensions also leads to results identical with those in the preceding section. It is in this sense that the treatment of the infrared modes as performed in the preceding section can be rated “universal”.

¹⁰Taking the Bessel functions and the n sum more accurately into account, the actual value of T_{cr} changes slightly: $T_{\text{cr}}/m_{\text{eff}} \simeq 2/21$.

6 Conclusions

In the present work, we have established and analyzed a dynamical model for the order parameter of the deconfinement phase transition in Yang-Mills theories – the vacuum expectation value of the Polyakov loop operator. We have calculated the effective action for this order parameter to second order in a derivative expansion, and have treated the gluonic fluctuations in one-loop approximation.

As a first conclusion, we observed that the “constant-Polyakov-loop” approximation, $A_0 = \text{const.}$, as considered in the literature, is in principle incapable of describing two different phases owing to the lack of an additional scale separating a high- and low-temperature phase in $d = 4$. This can also be inferred from the observation that the vacuum expectation value of the trace of the energy momentum tensor for a constant quasi-abelian A_μ background vanishes:

$$\langle T^\mu_\mu \rangle = \beta_{g^2} F_{\mu\nu} F_{\mu\nu} = 0, \quad \text{for } A_\mu^a = n^a A_\mu = \text{const.} \quad (53)$$

Therefore, a vacuum model of this type must necessarily preserve scale invariance even at finite temperature so that the theory must remain in a single phase.

In the present model, scale-breaking is induced by fluctuations of the Polyakov loop which, in a particular choice of gauge, are associated with a nonvanishing electric field. The question of whether or not these fluctuations are energetically favored can in principle be answered by the dynamics of the model. It turns out that the deconfinement phase is the generic phase in the absence of fluctuations (this holds for all $d \geq 4$). Whether spatial Polyakov loop fluctuations drive the model into a confining phase depends on the form of the weight function $W(c)$ of the kinetic term.

In four spacetime dimensions, thermal infrared singularities complicate the investigation of the weight function and require additional specifications of how to deal with these singularities. Within a regularization-independent scheme that introduces no other scale than already present, the $d = 4$ model does not reveal a confinement phase; instead, fluctuations of the Polyakov loop even favor a deconfining vacuum state.

By contrast, when regularizing the infrared by a physically effective cutoff comparable to a gluon mass for the transverse modes, a phase transition into a confining phase for low temperature becomes visible in the strong-coupling regime.

Whether one of these scenarios is realized in Yang-Mills theory cannot, of course, be answered within a perturbative approach like that employed in the present paper. Not only does the enormous extrapolation of a one-loop calculation into the strong-coupling sector present a major problem, but, with regard to the infrared singularities, (even nonperturbatively) integrating out the gluonic fluctuations at one fell swoop seems to be inappropriate. Instead, the integration over the fluctuations should be performed step by step in order to control a possible emergence of a dynamically generated mass scale.

The one-loop model at least facilitates a concrete investigation of possible scenarios, and at most displays some features in a qualitatively correct manner. An appraisal of the different scenarios requires further arguments. The first scenario of Sect. 4.1 without an effective mass can be preferred only from a theoretical viewpoint owing to its simplicity and

universality. Though the second scenario of Sect. 5 needs more input, the appearance of an additional infrared mass scale is common to almost all conjectured confining low-energy effective theories of Yang-Mills theory; therefore, a phenomenological viewpoint supports this scenario from the beginning, and so does the final result. Nevertheless, a reliable investigation of the infrared requires nonperturbative methods.

Let us finally comment on the differences of our results to Ref. [10] which inspired the model considered in the present work; although the representations of the effective action in the form of Eq. (6) are congruent, the meaning of the results is quite different in comparison: in [10], the fluctuations of the A_0 field have not been taken into account, implying that the resulting “effective action” remains a complete quantum theory of the A_0 field. The A_0 ground state is then *approximated* by the effective potential which is obtained by transforming the kinetic term to standard canonical form. By contrast, we integrated over *all* quantum fluctuations of the A_μ field in the present work; therefore, the resulting effective action is the generating functional of the 1PI diagrams and governs the dynamics of the background fields in the sense of classical field theory. To conclude, it is not astonishing that the explicit results of [10] in particular for the weight function $W(c)$ do not agree with ours because they have a different origin and a different meaning.¹¹

In spacetimes with more than four dimensions, the situation simplifies considerably: on the one hand, the model is infrared finite, thereby producing unambiguous results; on the other hand, there already exists another dimensionful scale given by the coupling constant. We discovered a phase transition from the (generic) deconfining to a confining phase for *increasing* temperature for $d > d_{\text{cr}} \simeq 7.42$. This is consistent with the fact that the dimensionless coupling constant grows for increasing energies, reaching an UV-stable fixed point. Beyond perturbation theory, the latter statement has also been confirmed in the nonperturbative framework of exact renormalization group flow equations [23].

Acknowledgments

The author wishes to thank W. Dittrich for helpful conversations and for carefully reading the manuscript. Furthermore, the author profited from insights provided by M. Engelhardt, whose useful comments on the manuscript are also gratefully acknowledged.

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¹¹From [10], it is in principle possible to arrive at our results (and get rid of the renormalization problems) by integrating over the A_0 fluctuations; however, for a consistent treatment, the second weight function of the $\mathcal{O}((\partial c)^4)$ term has to be known before integrating over the A_0 fluctuations.

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